Comparisons are made between modal synthesis approximations and reference solutions for system eigenvalues and eigenvectors. The complex eigenvalues are denoted by

$$\lambda = -\sigma + i\omega \tag{20}$$

and, in Table 1, the percentage error in σ and ω is given for three levels of truncation.

From the results in Table 1 it can be seen that a truncated set of free-interface substructure complex modes can be employed to represent a system with nonproportional damping by using the proposed first-order substructure coupling method. It is of interest to note that the frequency and damping errors are relatively large for many modes. This is probably due to the fact that only free-interface complex modes were employed. Also it is of interest to note that frequency and damping convergence is not necessarily monotonic (e.g., mode 3 damping) as in the familiar Rayleigh-Ritz approximation of frequencies of undamped systems.

Concluding Remarks

A generalized substructure coupling procedure for damped structures has been described, and a typical example of the effect of modal truncation on system eigenvalues has been presented. Further research on related coupling methods is currently in progress.

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Torsional Vibrations of Orthotropic Conical Shells

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Introduction

A SIMPLE model is presented for use by researchers in engineering and physical sciences working specifically on the design of modern missiles and space vehicles. With the advancement of space research, it has become necessary to obtain a greater insight into the behavior of shells of orthotropic material which are frequently used in missiles and other allied systems. In spite of the importance of the problem, little work has been done on such shells, although Hearmon and Mirsky[1] and Garnet et al.[2,3] have presented a study of torsional vibrations of orthotropic cylindrical shells.

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In this Note the differential equation governing the torsional vibrations of an orthotropic conical shell is obtained and solved by the Rayleigh-Ritz technique of assuming the displacements as an infinite series in terms of meridional coordinates. Numerical results for the frequency parameter have been computed for the first two modes of vibration for different values of length parameter, thickness parameter, and semivertical angle.

Deformations

Consider a conical shell of axial length ℓ, thickness h, and semivertical angle α and let R be the mean cross-sectional radius of the shell. Let the shell be referred to a coordinate system r, θ, z, where r is the radial coordinate measured from the middle surface of the shell along the outward drawn normal to the surface, θ the tangential direction, and z the meridional direction. Let u_r, u_θ, and u_z be the components of displacement in r, θ, and z directions, respectively. Since the torsional vibrations are being considered here for the shell,

$$u_r = 0 = u_z \text{ and } \partial(\ )/\partial\theta = 0 \tag{1}$$

The displacement u_θ is approximated by $\bar{u}_\theta$ as

$$\bar{u}_\theta = v + r\Psi_\theta \tag{2}$$

where v is the tangential displacement in the direction of θ of any particle which is lying on the middle surface of the shell and Ψ_θ is the rotation of a normal element in the $r\theta$ plane.

Moment and Force Resultants

Moment and force resultants as defined[4] reduce to

$$\begin{aligned} N_{z\theta} &= C_{66} h \left(\frac{\partial v}{\partial z} + \frac{h^2}{12 R_1} \frac{\partial \Psi_\theta}{\partial z} \right) \\ M_{z\theta} &= C_{66} \frac{h^3}{12} \left(\frac{1}{R_1} \frac{\partial v}{\partial z} + \frac{\partial \Psi_\theta}{\partial z} \right) \end{aligned} \tag{3}$$

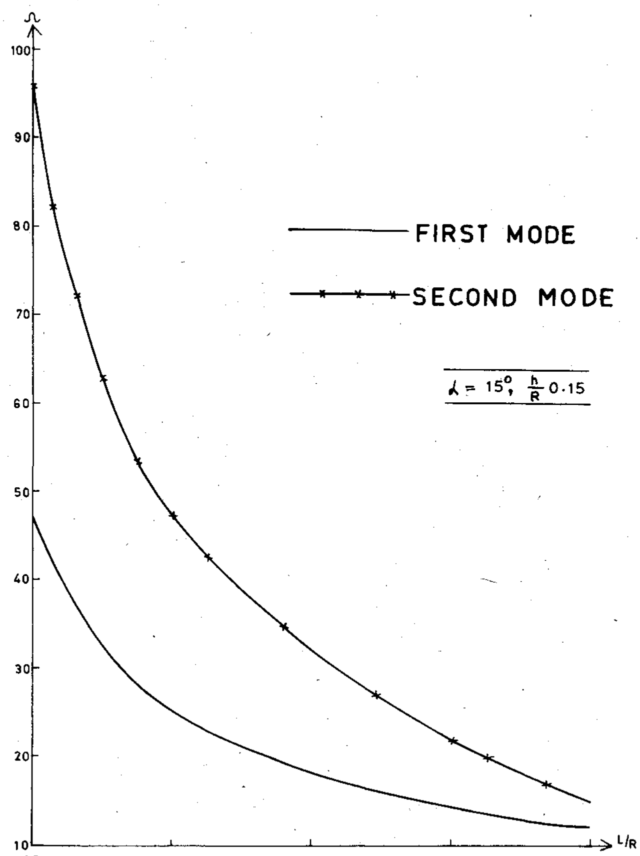

Fig. 1 Variation of frequency with length parameter for clamped orthotropic conical shells in the first two modes of vibration.

and

$$Q_\theta = \frac{C_{55}\eta h}{R_I K_s}\left(\Psi_\theta - \frac{v}{R_I}\right) \tag{4}$$

after making the substitution for shear stresses $\sigma_{z\theta}$ and $\sigma_{r\theta}$ and using the limitations of Eqs. (1) and (2) for shear strains $\epsilon_{z\theta}$ and $\epsilon_{r\theta}$.

Here $R_I = z\tan\alpha$, $\eta = (1 + h^2/12R_I^2)$ and factor K_s includes the effects due to shear. C_{55} and C_{66} are needed to characterize the elastic behavior of the orthotropic material of the shell.

Energy Considerations

The strain energy of the shell is given by

$$\bar{W} = \tfrac{1}{2}(\sigma_{z\theta}\epsilon_{z\theta} + \sigma_{r\theta}\epsilon_{r\theta}) \tag{5}$$

The total strain energy W of the conical shell is obtained by multiplying $\bar{W}$ with the element of volume $[(R_I + r)/R_I]z\sin\alpha \mathrm{d}r\mathrm{d}\theta\mathrm{d}z$ and then integrating over the volume of the shell, i.e.,

$$W = \pi\int_{z_I}^{z_2}\left[N_{z\theta}\frac{\partial v}{\partial z} + M_{z\theta}\frac{\partial \Psi_\theta}{\partial z} + Q_\theta\left(\Psi_\theta - \frac{v}{R_I}\right)\right]z\sin\alpha \mathrm{d}z \tag{6}$$

The first variation of the strain energy W is therefore given by

$$\delta W = 2\pi\int_{z_I}^{z_2}\left[N_{z\theta}\delta\left(\frac{\partial v}{\partial z}\right) + M_{z\theta}\delta\left(\frac{\partial \Psi_\theta}{\partial z}\right) + Q_\theta\delta\left(\Psi_\theta - \frac{v}{R_I}\right)\right]z\sin\alpha \mathrm{d}z \tag{7}$$

The kinetic energy $\bar{T}$ is given by

$$T = \tfrac{1}{2}\rho(\partial u_\theta/\partial t)^2$$

and the first variation of kinetic energy T is obtained as

$$\delta T = 2\pi\rho\int_{z_I}^{z_2}\left[R_I h\left(\frac{\partial v}{\partial t}\right)\delta\left(\frac{\partial v}{\partial t}\right) + \frac{h^3}{6}\left\{\frac{\partial v}{\partial t}\delta\left(\frac{\partial \Psi_\theta}{\partial t}\right) + \frac{\partial \Psi_\theta}{\partial t}\delta\left(\frac{\partial v}{\partial t}\right)\right\} + \frac{R_I h^3}{12}\frac{\partial \Psi_\theta}{\partial t}\delta\left(\frac{\partial \Psi_\theta}{\partial t}\right)\right]\cos\alpha \mathrm{d}z \tag{8}$$

Equation of Motion

Using Hamilton's energy equation

$$\int_{t_0}^{t_I}(\delta T - \delta W)\mathrm{d}t = 0 \tag{9}$$

and considering only the steady-state vibrations, the variational equation in terms of v and Ψ_θ comes out to be

$$\begin{aligned}&\int_{t_0}^{t_I}\int_{z_I}^{z_2}\left[\left\{R_I N'_{z\theta} + \tan\alpha N_{z\theta} + Q_\theta - \rho R_I h\ddot{v} - \frac{h^3}{6}\rho\ddot{\Psi}_\theta\right\}\delta v + \left\{R_I M'_{z\theta} + \tan\alpha M_{z\theta} - R_I Q_\theta - \frac{h^3}{12}\rho(2\ddot{v} + R_I\ddot{\Psi}_\theta)\right\}\delta\Psi_\theta\right]\\ &\times\cos\alpha \mathrm{d}t\mathrm{d}z - \int_{t_0}^{t_I}\left[zN_{z\theta}\delta v + zM_{z\theta}\delta\Psi_\theta\right]_{z_I}^{z_2}\sin\alpha \mathrm{d}t\\ &+\rho\int_{z_I}^{z_2}\left[R_I h\dot{v} + \frac{h^3}{6}\dot{\Psi}_\theta\delta v + \frac{h^3}{12}(2\dot{v} + R_I\dot{\Psi}_\theta)\delta\Psi_\theta\right]_{t_0}^{t_I}\mathrm{d}z = 0\end{aligned} \tag{10}$$

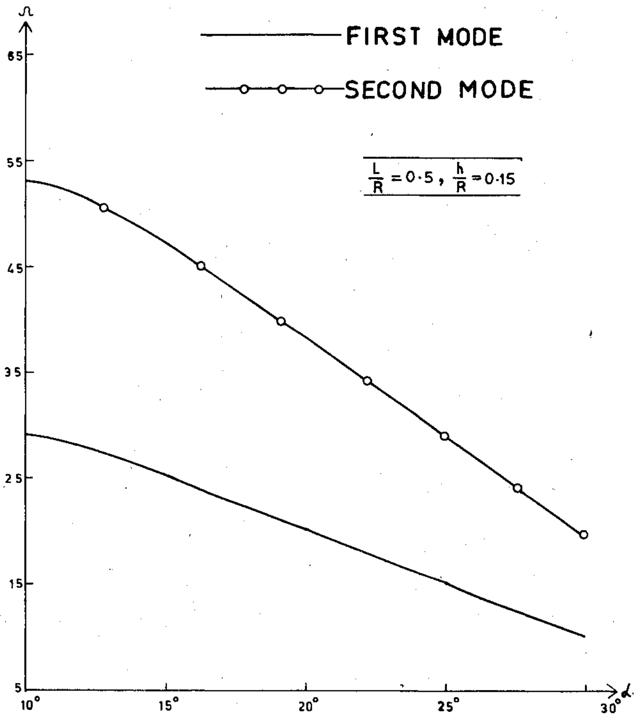

Fig. 2 Variation of frequency with semivertical angle for clamped orthotropic conical shells in the first two modes of vibration.

where dashes denote the partial differentiation with respect to z and dots the partial differentiation with respect to t.

Boundary Conditions

The appropriate boundary conditions obtained at the two edges are

$$v = 0 \text{ and } \Psi_\theta = 0 \text{ at the edges, } \quad z = z_I \text{ and } z_2 \tag{11}$$

if the edges of the shell are clamped, and

$$N_{z\theta} = 0 \text{ and } M_{z\theta} = 0 \text{ at } z = z_I \text{ and } z_2 \tag{12}$$

if the edges of the shell are free.

We shall, however, consider the vibrations of the clamped shell only. We use the following transformations of coordinates,

$$z = z_I e^x \tag{13}$$

so that at

$$z = z_I,\ x = 0;\ z = z_2,\ x = x_2,\ \text{where } x_2 = \log_e(z_2/z_I) \tag{14}$$

Method of Solution

For free harmonic wave propagation in a conical shell clamped at the two edges $z = z_I$ and z_2, the solution for the displacements is assumed to be

$$v = \sum_{m=1}^{\infty} A_m \sin\xi_m x\cos pt$$

$$\Psi_\theta = \sum_{n=1}^{\infty} B_n \sin\xi_n x\cos pt \tag{15}$$

where p is the circular frequency of vibration of the shell and $\xi_k = k\pi/12$.

Substituting the moment and force resultants in terms of the new variable x and v and Ψ_θ from Eqs. (15) into Eq. (10)

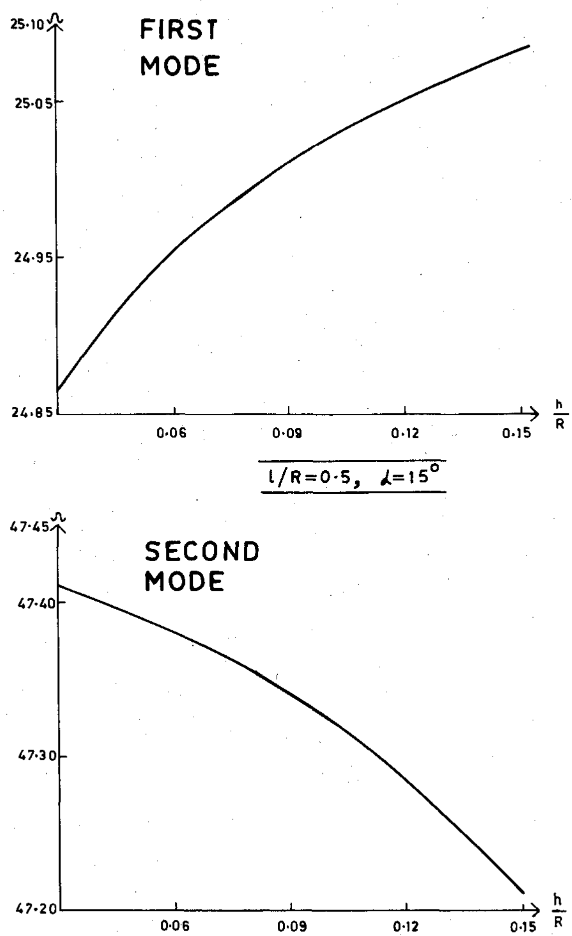

Fig. 3 Variation of frequency with the thickness parameter for clamped conical shells.

and making use of the transformation of Eq. (13) and the boundary conditions [Eq. (12)], we finally get the following variational equation

$$\sum_{m=1}^{\infty} f_m(\theta)\delta A_m + \sum_{n=1}^{\infty} f_n(\phi)\delta B_n = 0 \tag{16}$$

where $f_m(\theta)$ and $f_n(\phi)$ are functions of α, C_{55}, C_{66}, K_s, h, Ω, etc.

Frequency Equation

The variational equation (16) should be satisfied for all arbitrary values of δA_1, $\delta A_2, \ldots$, δB_1, $\delta B_2, \ldots$, etc. Therefore, the coefficients of these quantities in Eq. (16) should all vanish separately, i.e.,

$$\begin{aligned} f_1(\theta) &= 0, \qquad f_2(\theta) = 0 \\ f_1(\phi) &= 0, \qquad f_2(\phi) = 0 \end{aligned} \tag{17}$$

By eliminating the unknown constants A_1, $A_2, \ldots$, B_1, $B_2, \ldots$, etc. from Eqs. (17), we obtain the desired frequency equation as

$$|S_{iJ}| = 0 \tag{18}$$

where $i,J = 1,2,\ldots$ and S_{iJ} are the coefficients of A_1, $A_2, \ldots$, B_1, $B_2, \ldots$, etc. in Eqs. (17). These equations are solved for the determination of the frequency parameter Ω by the Rayleigh-Ritz iteration procedure.

Numerical Results and Discussion

The numerical results for the frequency parameter Ω have been computed for the first two modes of torsional vibration of orthotropic conical shells clamped at the two edges for various values of length parameter ℓ/R, thickness parameter h/R, and semivertical angle α. The numerical values of the elastic constants are taken from Ref. 4. These are $C_{55} = 0.1678$, $C_{66} = 0.1658$, and $1/K_s = 1.19$.

The frequency curves (frequency vs length parameter, frequency vs thickness parameter, and frequency vs semivertical angle) are plotted in Figs. 1-3 corresponding to the first two modes of vibration for clamped conical shells. Figure 1 shows that the frequency parameter decreases with the increase in ℓ/R, first slowly and then rapidly for both the modes of vibration. This variation of frequency for the second mode of vibration is higher than for the first. Figure 2 shows that the frequency parameter also decreases with the increase in semivertical angle α for the two modes of vibration, but here the frequency parameter decreases first rapidly and then slowly. Also the frequency decreases asymptotically with higher values of α for both modes of vibration. Figure 3 indicates the interesting behavior of the frequency parameter Ω with the increase in thickness parameter h/R. Ω increases with the increase in h/R very slightly in the first mode of vibration, whereas it decreases very slightly in the second mode of vibration. Therefore, the frequency of torsional vibrations of the orthotropic conical shell is not sensitive to the thickness of the shell.

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